Propagation of Quasiplane Acoustic Waves along an Impedance Boundary

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The parabolic approximation for the acoustic equations of motion is applied to the study of the sound field generated by a time harmonic plane wave at grazing incidence to a finite impedance boundary. The resulting equations possess a solution that may be expressed in terms of the complementary error function. Asymptotic expansion of this solution for field points near the boundary provides results compatible with those for a point source on the boundary for both the soft boundary (finite impedance) and hard boundary (the limit in which the impedance becomes infinite) cases. The presence of a surface wave in the solution is also established.

Introduction

A THOROUGH understanding of the propagation of acoustic disturbances in the near vicinity of an absorbing plane is of fundamental importance for many practical applications of acoustic theory. The most elementary example is provided by the consideration of the reflection of plane waves from a locally reacting plane of impedance ζ. For all incidence angles except grazing, the solution of this problem is well known.[1] Equally well known is the failure of the analysis to provide any insight into the behavior of the acoustic field for grazing incidence.[2-4] It was recognized quite early that, at grazing, "a different mathematical analysis is required in order to describe the curvature of the otherwise plane wave front introduced by its propagation over an absorbing sample."[5] Interestingly, even though this suggests that wave front curvature is introduced by the impedance plane, all analyses that have attempted to determine the acoustic field at or near grazing incidence have been of point sources.[6-9] Here, wave front curvature is introduced by the source as well as by the boundary.

Of course, plane waves are a mathematical abstraction and cannot be produced physically. Every real source is, after all, three-dimensional. Thus, the many theoretical analyses[6-9] that have been devoted to explaining the essential features of the acoustic field of a point source near an impedance plane may be considered as constituting a sufficient body of knowledge for any practical application. Nevertheless, there is something intrinsically unsatisfactory in being unable to explain what should be the most elementary case—the grazing incidence plane wave—when one has a full explanation of the most complete case—the field from a point source—especially when the solution of the plane wave problem should be easier to obtain. Furthermore, there is no reason to suspect that any of the physical phenomena apparent in the fully three-dimensional field from a point source, with the obvious exception of geometric spreading, should be absent from a properly generalized plane wave analysis.

It is the primary purpose of this paper to derive an analytical solution that elucidates the essential features associated with the propagation of an initially plane wave at grazing incidence to a surface of finite impedance. It is shown that, with the exception of those characteristics dependent solely on the geometry, all essential properties normally attributed to the field of a point source near a finite impedance plane may be imputed to the field of an initially plane wave at grazing incidence. Noticeably absent is the paradoxical cancellation of the incident disturbance by the reflected field that is traditionally associated with this problem.

Analytical Solution

Problem Formulation

It is difficult to believe that the essential physical phenomena associated with acoustic disturbances propagating in the near vicinity of an absorbing plane are intrinsically related to the curvature of the incident wave front. Clearly, if the form of the solution is properly generalized, the effects of an absorbing boundary on an initially plane wave at grazing incidence must be amenable to analysis. Two interrelated features of the usual proposed form of the solution for plane waves at grazing incidence may be considered as leading to its failure. The first is the fact that the traditional analysis does not allow variation of the total field in the direction normal to the absorbing boundary. However, a component of the acoustic velocity vector normal to the surface is required in order to satisfy the impedance boundary condition. These two conditions are incompatible unless the total acoustic pressure is zero at the boundary and, hence, throughout the field. Thus, it is to be expected that the proper generalization of the analysis for grazing incidence will allow the acoustic field to vary in the direction normal to the absorbing plane. Note, in particular, that this generalization places no restrictions on the incident waveform. For the purposes of the current analysis, the incident wave will be assumed to be a time harmonic plane wave.

The second feature of the customary solution that leads to its ultimate failure at grazing incidence is the fact that the proposed solution form allows no variation of the total field in the propagation direction other than that resulting from

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translation of a fixed waveform at sonic velocity along the boundary. Thus, no measure of distance from the "source" is introduced. It is clear that the waveform cannot remain unchanged with propagation distance if the boundary is to have any effect on the field. Thus, the proper generalization of the analysis for grazing incidence must also allow for variation of the field with propagation distance.

In the analysis to follow, both of these shortcomings of the traditional plane wave solution form are avoided by considering quasiplane waves. These may be defined as waves that, to leading order, may be expressed in terms of plane wave fronts. However, both amplitude and phase are allowed to vary along these fronts in a manner reminiscent of the behavior of the evanescent waves that arise in certain transmission-diffraction problems. In the approximate theory to be developed, these variations are assumed to be slow in the sense that they occur over distances which are large compared to a wavelength.

For the purposes of the current discussion, consideration will be limited to two-dimensional time harmonic disturbances, of angular frequency ω, propagating in an ideal gas translating at uniform velocity $cM\hat{x}$. Here, c is the sound speed, $\hat{x}$ a unit vector directed along the positive x axis, and M the mean flow Mach number. The medium occupies the half-space above an absorbing plane boundary, of specific acoustic impedance ζ, which lies on $y = 0$. The field is to be determined in $x > 0$, $y > 0$. A common factor $e^{-i\omega t}$ is suppressed in all functions describing field quantities. It is assumed that the acoustic pressure at $x = 0$ is given by

$$p(0,y) = 1 \qquad y > 0 \tag{1}$$

and that a radiation condition is to be satisfied for $x \to +\infty$. Consideration of the form of a pure plane wave propagating in the x direction suggests that it is reasonable to seek a solution for $p(x,y)$ in the form

$$p = A(x,y)\exp[ikx/(1+M)] \tag{2}$$

with $k = \omega/c$. Then, upon substituting Eq. (2) into the well-known convected wave equation, it follows that $A(x,y)$ must satisfy[10]

$$2ikA_x + A_{yy} + (1-M^2)A_{xx} = 0 \qquad (x,y > 0) \tag{3}$$

The impedance condition $p = \rho c\zeta(1+M)V$, where V is the y component of the acoustic velocity and ρ the ambient density, is imposed along $y = 0$. Use of the y component of the linearized momentum equation

$$\rho c(-ikV + MV_x) + P_y = 0$$

then leads to the boundary condition on $A(x,y)$ in the form

$$A(x,0) + \frac{1}{ik\mu}A_y(x,0) - \frac{M(1+M)}{ik}A_x(x,0) = 0 \tag{4}$$

in which the symbol

$$\mu = 1/[(1+M)^2\zeta] \tag{5}$$

has been introduced. In addition, Eq. (1) requires that

$$A(0,y) = 1 \tag{6}$$

The analysis of the problem will be carried out by application of the parabolic approximation. This approximation is obtained from the above equations by neglecting the last term in both the differential equation (3) and the impedance boundary condition [Eq. (4)]. This provides the equations

$$2ikA_x + A_{yy} = 0 \qquad (x,y > 0) \tag{7}$$

$$A_y(x,0) + ik\mu A(x,0) = 0 \qquad (x > 0) \tag{8}$$

The parabolic approximation will not be justified in detail here. It is widely used in underwater acoustics studies[11] and has been developed for atmospheric propagation.[12] Further, the final results of the current study suggest that it is fully justified in the present application. In general, the approximation given by Eqs. (7) and (8) is based on the assumption that variations in $A(x,y)$ along x occur over distances that are large compared to a wavelength. The approximation is commonly applied to simplify numerical analysis. In the following, however, an analytical solution of the current problem will be obtained.

Solution Procedure

The problem posed, consisting of Eq. (7) and auxiliary conditions of Eqs. (6) and (8), is an initial boundary value problem for the parabolic equation (7) in which x is the time-like variable. The boundary condition [Eq. (8)] is of the third kind. Since the coefficients of Eqs. (7) and (8) are constant, any derivative of a solution also satisfies the equations. Thus, the function

$$B = A - 1 + (1/ik\mu)A_y \tag{9}$$

must satisfy Eq. (7) if A does. By means of this change of dependent variable, Eqs. (6–8) can be transformed to the more tractable problem,

$$2ikB_x + B_{yy} = 0 \qquad (x,y > 0) \tag{10}$$

$$B(0,y) = 0 \qquad (y > 0) \tag{11}$$

$$B(x,0) = -1 \qquad (x > 0) \tag{12}$$

which, except for the complex coefficient, is the same as the Rayleigh problem for the suddenly accelerated flat plate in a viscous fluid[13] or, with $T(x,y) = B + 1$, the problem for heat transfer to a semi-infinite wall.[14] These problems possess similarity solutions in terms of the error function

$$\operatorname{erf}(\eta) = \frac{2}{\sqrt{\pi}}\int_0^{\eta} e^{-t^2}\,\mathrm{d}t \tag{13}$$

with $\eta = \eta(x,y)$. The current problem, then, may be expected to possess a similarity solution also. The complex coefficient $2ik$ suggests that this solution will be expressible in terms of the Fresnel integrals

$$\hat{C}(\eta) = \int_0^{\eta} \exp(i\pi t^2/2)\,\mathrm{d}t = C(\eta) + iS(\eta) \tag{14}$$

In fact, if $B(x,y)$ is sought in the form

$$B = F(\eta) \tag{15}$$

where η is the single independent variable

$$\eta = (k/\pi x)^{\frac{1}{2}}y \tag{16}$$

then straightforward algebraic manipulation shows that Eq. (10) is equivalent to

$$F'' - i\pi\eta F' = 0 \tag{17}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}\eta}[F'\exp(-i\pi\eta^2/2)] = 0 \tag{18}$$

Two elementary integrations and application of the auxiliary conditions [Eqs. (11) and (12)] then yield

$$F(\eta) = (1-i)\hat{C}(\eta) - 1 \tag{19}$$

The solution $A(x,y)$ is now determined by integration of Eq. (9), which may be considered as a first-order ordinary differential equation for A as a function of η with x as a parameter. It may be written as

$$A_\eta + i\pi\xi A = i\pi\xi(1-i)\hat{C}(\eta) \tag{20}$$

in which the parameter ξ has been introduced for convenience as

$$\xi = \mu(kx/\pi)^{\frac{1}{2}} \tag{21}$$

Equation (20) is equivalent to

$$\frac{\partial}{\partial\eta}[A\ \exp(i\pi\xi\eta)] = i\pi\xi\ \exp(i\pi\xi\eta)(1-i)\hat{C}(\eta) \tag{22}$$

Now the right side of Eq. (22) is

$$(1-i)\hat{C}(\eta)\frac{\partial}{\partial\eta}\exp(i\pi\xi\eta) = (1-i)\frac{\partial}{\partial\eta}[\hat{C}(\eta)\exp(i\pi\xi\eta)] - (1-i)\ \exp\left[\frac{i\pi}{2}(\eta^2+2\xi\eta)\right] \tag{23}$$

in which the fact that, from Eq. (14),

$$\frac{\mathrm{d}\hat{C}}{\mathrm{d}\eta} = \exp\left(\frac{i\pi\eta^2}{2}\right) \tag{24}$$

has been used. Employing Eq. (24) again shows that the right side of Eq. (23) is

$$(1-i)\frac{\partial}{\partial\eta}\left[\hat{C}(\eta)\exp(i\pi\xi\eta) - \hat{C}(\eta+\xi)\exp\left(-\frac{i\pi\xi^2}{2}\right)\right] \tag{25}$$

Therefore, the solution A follows directly by an integration of Eq. (22) as

$$A = (1-i)\hat{C}(\eta) - (1-i)\hat{C}(\eta+\xi)\exp(-i\pi\xi\eta - i\pi\xi^2/2) + G(\xi)\exp(-i\pi\xi\eta) \tag{26}$$

where the function $G(\xi)$ is arbitrary insofar as Eq. (20) is concerned. However, as $x \to 0$, $\eta \to \infty$ and $\hat{C}(\eta) \to (1+i)/2$. Thus, in order for A to satisfy the boundary condition [Eq. (6)], Eq. (26) implies that

$$A(0,y) = 1 = 1 - \exp(-i\mu ky) + G(0)\exp(-i\mu ky) \tag{27}$$

or that

$$G(0) = 1 \tag{28}$$

In fact, however, $G(\xi)$ is not arbitrary, because A must satisfy Eq. (7) and all of the auxiliary conditions on A have been applied. Thus, substitution of Eq. (26) into Eq. (7) should determine G. Carrying out the substitution shows that G must satisfy

$$G'(\xi) + i\pi\xi G = 0 \tag{29}$$

which, with Eq. (28), yields

$$G(\xi) = \exp(-i\pi\xi^2/2) \tag{30}$$

Finally, then, Eq. (26), which is the complete analytical solution to the problem posed in Eqs. (6–8), becomes

$$A(x,y) = (1-i)\hat{C}(\eta) - \exp(-i\pi\xi\eta - i\pi\xi^2/2)[(1-i)\hat{C}(\eta+\xi) - 1] \tag{31}$$

For purposes of later discussion, it is convenient to write the result of Eq. (31) in an alternate form using the identity[15]

$$\hat{C}(z) = \frac{1+i}{2}\mathrm{erf}\left[\frac{\sqrt{\pi}}{2}(1-i)z\right] = \frac{1+i}{2}\mathrm{erf}\left[\left(-\frac{i\pi}{2}\right)^{\frac{1}{2}}z\right] \tag{32}$$

Thus,

$$1-(1-i)\hat{C}(\eta) = 1 - \mathrm{erf}[(-i\pi/2)^{\frac{1}{2}}\eta] \equiv \mathrm{erfc}[(-i\pi/2)^{\frac{1}{2}}\eta] \tag{33}$$

so that A may be expressed as

$$A = 1 - \exp\left(\frac{i\pi\eta^2}{2}\right)\left\{\exp\left(-\frac{i\pi\eta^2}{2}\right)\mathrm{erfc}\left[\left(-\frac{i\pi}{2}\right)^{\frac{1}{2}}\eta\right] - \exp\left[-\frac{i\pi(\eta+\xi)^2}{2}\right]\mathrm{erfc}\left[\left(-\frac{i\pi}{2}\right)^{\frac{1}{2}}(\eta+\xi)\right]\right\} \tag{34}$$

Finally, if the definition[15]

$$w(z) = \exp(-z^2)\ \mathrm{erfc}(-iz) \tag{35}$$

is introduced, A assumes perhaps its simplest form,

$$A = 1 - \exp(i\pi\eta^2/2)\{w[(i\pi/2)^{\frac{1}{2}}\eta] - w[(i\pi/2)^{\frac{1}{2}}(\eta+\xi)]\} \tag{36}$$

It is noted that the argument in the last term of Eq. (36) is

$$\left(\frac{i\pi}{2}\right)^{\frac{1}{2}}\left[\left(\frac{k}{\pi x}\right)^{\frac{1}{2}}y + \left(\frac{kx}{\pi}\right)^{\frac{1}{2}}\mu\right] = \left(\frac{ikx}{2}\right)^{\frac{1}{2}}\left(\frac{y}{x}+\mu\right) \tag{37}$$

which is similar to the so-called numerical distance used in many analyses of sound propagation near an absorbing boundary.[16,17]

Discussion

In the representation expressed by Eq. (36), the first term represents the incident wave. It is convenient to call the quantity $(A-1)$ the radiated wave. Since the variable η is independent of the boundary properties, the effect of the boundary impedance ζ is contained completely in the final term. As will be shown later, this term also contains the surface wave. Note that, at $y = 0$, the first two terms of Eq. (36) cancel and

$$|A| = |w[(ikx/2)^{\frac{1}{2}}\mu]| \tag{38}$$

which gives the magnitude of the acoustic pressure along the boundary.

Insight into the behavior of the solution may also be obtained by considering the impedance plane as if it were a shadow boundary similar to that which occurs when a plane wave is incident on a knife-edge.[17,18] The second term of Eq. (36) is exactly the analytical description of the acoustic field on the dark side of the shadow boundary. The interaction of the incident wave with this shadow boundary field produces the diffraction bands that are characteristic of the illuminated region near the shadow boundary.

Thus, the three terms of Eq. (36) represent, respectively, the incident disturbance, a general diffracted field that cancels the incident field at the boundary, and a boundary correction term containing all of the impedance effects. Note, in fact, that the field produced solely by the first two terms of Eq. (36) cannot transfer energy from the acoustic field to the boundary, because the pressure associated with these two terms is identically zero there. This should be expected, because neither of these terms depends on the boundary impedance and no further terms independent of the impedance are part of the solution. The energy associated with these two terms must, ultimately, be absorbed by the impedance plane. This occurs because they produce a nonzero acoustic particle velocity normal to the boundary. The required energy transfer arises from the interaction of this velocity with the pressure arising from the third term of Eq. (36). As is easily verified, for fixed $y < \infty$, $A \to 0$ as $x \to \infty$.

Asymptotic Forms

Asymptotic representations of the solution for the cases of interest are most easily accomplished by introduction of the scaled variables

$$X = kx|\mu|^2 \quad (39)$$

$$Y = ky|\mu| \quad (40)$$

with

$$\mu = |\mu|(\cos\sigma - i\sin\sigma) \quad (41)$$

where σ is the phase angle of the boundary impedance. Since for a realistic boundary the real part of the impedance is positive, σ must satisfy the inequality

$$-\pi/2 < \sigma < \pi/2 \quad (42)$$

Employing Eqs. (39) and (40) in Eqs. (16) and (21) shows that

$$\xi = (X/\pi)^{\frac{1}{2}} e^{-i\sigma} \quad (43)$$

$$\eta = Y/(\pi X)^{\frac{1}{2}} \quad (44)$$

Thus, Eq. (36) may be written as

$$A = 1 - \exp(iY^2/2X)\{w[(i/2X)^{\frac{1}{2}}Y] - w[(iX/2)^{\frac{1}{2}}(Y/X + e^{-i\sigma})]\} \quad (45)$$

a form indicating that the magnitude of the boundary impedance affects the solution in a manner which is scaled out of the solution if the variables X and Y are used. On the other hand, the phase of the impedance affects the solution in a more fundamental way since it remains as a parameter in Eq. (45).

Soft-Boundary Case

The soft-boundary case is defined by the inequalities[9,16]

$$X^{\frac{1}{2}} \gg 1$$

$$Y^2/X \ll 1 \quad (46)$$

which restrict the observation point (x,y) to lie in a region where the impedance boundary has had an effect on the wave.

Under these conditions, a small argument expansion is required for the second term in Eq. (45); to leading order, it is[15]

$$w[(i/2X)^{\frac{1}{2}}Y]\exp(iY^2/2X) \approx (2/i\pi X)^{\frac{1}{2}}Y \quad (47)$$

On the other hand, the argument of the last term in Eq. (45) is large in the soft-boundary limit. This term then requires use of the asymptotic expansion of $w(z)$ for large $|z|$. This is

$$w(z) = e^{-z^2}\operatorname{erfc}(-iz) \approx 2H[-\operatorname{Im}(z)]e^{-z^2} + i\left(1 + \frac{1}{2z^2}\right)\Big/ z\pi^{\frac{1}{2}} \quad (48)$$

where $H(s)$ is the unit step function

$$H(s) = 0 \quad s < 0$$
$$= 1 \quad s > 0 \quad (49)$$

For the current application

$$z = (iX/2)^{\frac{1}{2}}(Y/X + e^{-i\sigma}) \quad (50)$$

and the term $2H[-\operatorname{Im}(z)]e^{-z^2}$ represents the surface wave. It is present in the field only for $\operatorname{Im}(z) < 0$, or for

$$\sin\sigma - \cos\sigma > Y/X \quad (51)$$

Now use of Eq. (50) under the restrictions given in the inequality (46) yields

$$1/z \approx (2/iX)^{\frac{1}{2}} e^{i\sigma}(1 - Y/Xe^{i\sigma}) + \mathcal{O}(X^{-2}) \quad (52)$$

$$\frac{1}{2z^2} \approx e^{2i\sigma}/iX + \mathcal{O}(X^{-2}) \quad (53)$$

In addition, for the purpose of comparison of the current analysis with that of Wenzel,[9] the quantity

$$\gamma = ik\mu = ik/(1+M)^2\zeta \quad (54)$$

is introduced. Wenzel's γ will be called γ_0 here, because in his work the ambient medium is stationary, with $M = 0$. Further, the expressions will be written in terms of the physical variables x and y to facilitate comparisons.

Finally, introduction of the results of Eqs. (47) and (48) into Eq. (45) and use of Eqs. (52–54) leads to the soft-boundary approximation for A in the form

$$A = \left(\frac{2i}{\pi kx}\right)^{\frac{1}{2}}\left(\frac{ik}{\gamma}\right)\left\{(1-\gamma y) + \frac{ik}{\gamma^2 x}\left[(1-\gamma y) + \frac{\gamma^2 y^2}{2}\right] + 2\exp\left(-\frac{i\gamma^2 x}{2k} - \gamma y\right)H\left(\cos\sigma - \sin\sigma - \frac{y}{|\mu|x}\right)\right\} \quad (55)$$

Equation (55) is closely comparable to Wenzel's Eq. (28) for the soft-boundary case. That equation expresses what Wenzel calls the "radiated" wave, which is the total field less both the incident wave and the reflected wave for a rigid boundary. In addition, Wenzel has written his equation without the surface wave term and his solution is for a point source. If the incident field is subtracted from Eq. (55) and if the surface wave is ignored, then the "radiated" field of the current analysis follows from Eq. (55) as

$$A_r = -\left\{1 + \left(\frac{ik}{\gamma^2 x}\right)^{\frac{1}{2}}\left(\frac{1}{i}\right)\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \times\left[(1-\gamma y) + \left(\frac{ik}{\gamma^2 x}\right)\left(1 - \gamma y + \frac{\gamma^2 y^2}{2}\right)\right]\right\} \quad (56)$$

On the other hand, Wenzel's Eq. (28) is

$$L = -\left[1 + \left(\frac{ik}{\gamma_0^2 r}\right)\left(1 - \gamma_0 h + \frac{\gamma_0^2 h^2}{2}\right)\right]\frac{e^{ikr}}{2\pi r} \quad (57)$$

For a source on the boundary, Wenzel's h is exactly the y of the current analysis and his r corresponds exactly to x. It should be noted that the factor $\exp[ikx/(1+M)]$ has been suppressed in Eq. (56). The variation of the field as described by the current analysis as a function of y at a fixed x is essentially identical to that of Wenzel's result.

In the far field where $kx \gg 1$, but with y still in accord with inequality (46), which is $ky^2/x \ll 1$ in physical variables, the total field less the surface wave term, is given by Eq. (56) as

$$A = \left(\frac{2i}{\pi kx}\right)^{\frac{1}{2}}\frac{ik}{\gamma}(1-\gamma y) \quad (58)$$

On the other hand, Wenzel's solution in the same limit gives

$$L_t = \left(\frac{ik}{\gamma_0^2 r}\right)(1-\gamma_0 h)\frac{e^{ikr}}{2\pi r} \quad (59)$$

which is his Eq. (36). For fixed x or r, these equations show that the behavior of the field as a function of distance above the boundary is described by the term

$$g(y) = 1 - \gamma y \tag{60}$$

The magnitude of the pressure, as a function of y, is then governed by

$$|g(y)| = (1 - 2|\gamma| y \sin\sigma + |\gamma|^2 y^2)^{\frac{1}{2}} \tag{61}$$

Thus, if $\sin\sigma > 0$, the amplitude of the wave decreases with increasing y until it reaches a minimum value at

$$ky = |\zeta|(1 + M)^2 \sin\sigma \tag{62}$$

[See Eqs. (5) and (54).] For $M = 0$, this equation is identical to Eq. (37) of Wenzel's paper. The variation of the phase of the wave with y is discussed in detail by Wenzel and will not be considered further here. Clearly, the two analyses lead to the same conclusions.

Soft-Boundary Surface Wave

Equation (51) defines the surface wave region and may be written in either of the two forms

$$Y < (\sin\sigma - \cos\sigma)X \tag{63a}$$

$$\cos\sigma < \sin\sigma - Y/X \tag{63b}$$

In dimensional variables, these inequalities become

$$y < (\alpha - \beta)x \tag{64a}$$

$$\beta < \alpha - y/x \tag{64b}$$

where

$$\alpha = |\mu| \sin\sigma, \qquad \beta = |\mu| \cos\sigma \tag{65}$$

The inequality (64a) simply states that the surface wave is restricted to the wedge-shaped region between the boundary and a line through the origin with slope $(\alpha - \beta)$. Clearly, a surface wave can exist only for

$$\pi/4 \leq \sigma < \pi/2 \tag{66}$$

since otherwise $(\alpha - \beta) < 0$. (Recall that the range of σ is $-\pi/2 < \sigma < \pi/2$.) This situation is illustrated in Fig. 1a. The inequality (64b) is a statement on the surface wave region in the (α,β) plane and is exhibited in Fig. 1b. For a given field point (x,y), the quantity y/x is fixed and a surface wave can exist at that point only if the impedance is such that this second inequality is satisfied. Note, however, that the inequality is meaningful only in conjunction with the inequalities (46) under which Eq. (55) was derived. As illustrated in Fig. 1c, these are often the most restrictive.

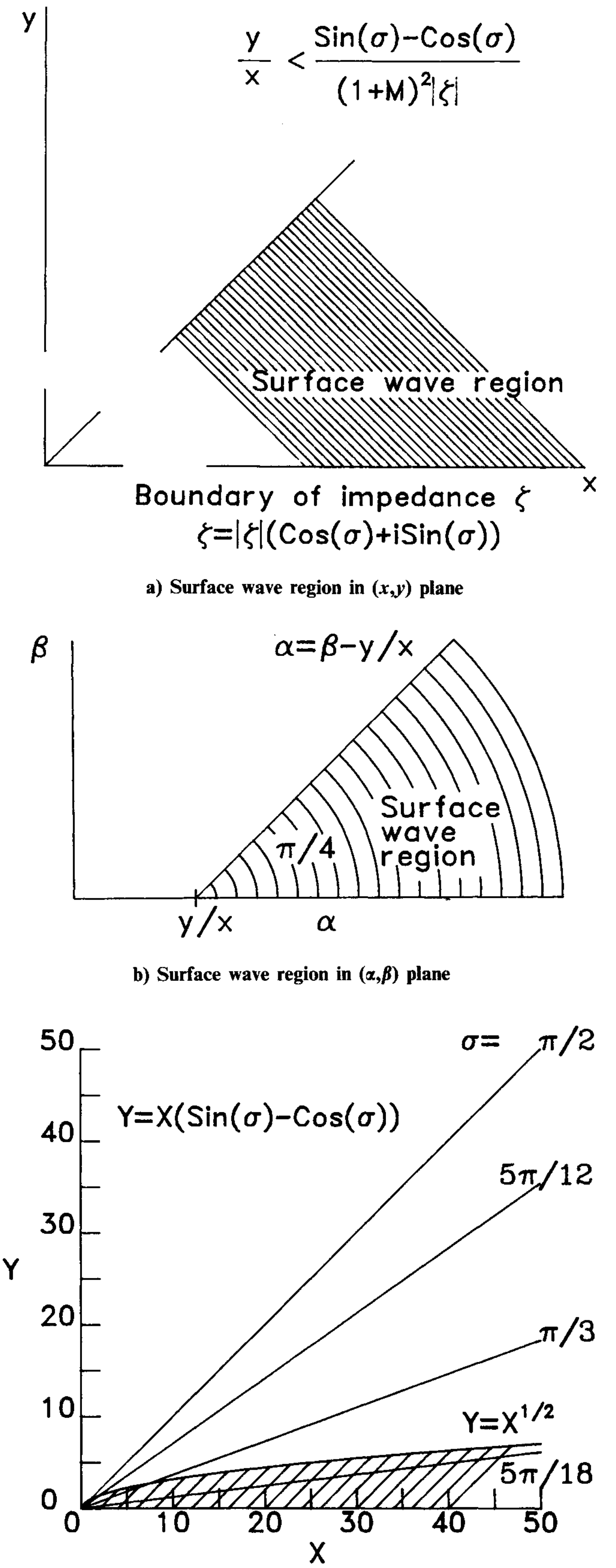
A fundamental characteristic of the surface wave is its exponential decay not only for increasing distance from the boundary, but also for increasing distance in the propagation direction. For the current situation, this decay is given by the factor

$$D_y = \exp(-ky|\mu| \cos\sigma) \tag{67}$$

in the y direction and

$$D_x = \exp(-kx|\mu|^2 \sin\sigma) \tag{68}$$

in the x direction. Inequality (66) indicates that both $\cos\sigma$ and $\sin\sigma$ are positive for all σ for which the surface wave occurs. As is well known, the surface wave in the soft-boundary case is generally a negligible part of the overall field unless $\sigma \approx \pi/2$, so that the decay of the wave with propagation distance is small. However, in this case, the surface wave is restricted to a region very near the boundary because $\cos\sigma \approx 1$.



a) Surface wave region in (x,y) plane

b) Surface wave region in (α,β) plane

c) Region of validity of asymptotic formulas, lower part of shaded region, compared with surface wave region.

Fig. 1 Summary of results for the soft-boundary surface wave.

Hard-Boundary Case

The hard-boundary case is defined by[9,16]

$$Y \ll 1, \qquad X^{\frac{1}{2}} \ll 1 \tag{69}$$

The most convenient form for asymptotic analysis is then given by use of Eqs. (43) and (44) in Eq. (34). This places $A(X,Y)$ in the form

$$\begin{aligned} A = 1 &- \operatorname{erfc}\left[\frac{Y}{\sqrt{2iX}}\right] \\ &+ \exp\left[-ie^{-i\sigma}\left(Y + \frac{e^{-i\sigma}X}{2}\right)\right] \\ &\times \operatorname{erfc}\left[\frac{Y}{(2iX)^{1/2}} - ie^{-i\sigma}\left(\frac{iX}{2}\right)^{1/2}\right] \end{aligned} \tag{70}$$

Then, to second order in $X^{\frac{1}{2}}$ and Y

$$\exp[-ie^{-i\sigma}(Y + e^{-i\sigma}X/2)] \approx 1 - ie^{-i\sigma}Y \tag{71}$$

and

$$\begin{aligned} &\operatorname{erfc}\left[\frac{Y}{(2iX)^{1/2}} - ie^{-i\sigma}\left(\frac{iX}{2}\right)^{1/2}\right] \\ &\approx \operatorname{erfc}\left[\frac{Y}{(2iX)^{1/2}}\right] - e^{-i\sigma}\left(\frac{2X}{i\pi}\right)^{\frac{1}{2}} \exp\left(\frac{iY^2}{2X}\right) \end{aligned} \tag{72}$$

To this order, then,

$$\begin{aligned} A(x,y) \approx (1 - \gamma y) &+ \gamma y \operatorname{erf}\left[\frac{ky}{(2ikx)^{1/2}}\right] \\ &+ \gamma e^{i\pi/4}\left(\frac{2x}{k\pi}\right)^{\frac{1}{2}} \exp\left(\frac{iky^2}{2x}\right) \end{aligned} \tag{73}$$

when the physical variables are used. The term $(1 - \gamma y)$ represents the surface wave.

Note that this expression for $A(x,y)$ cannot be valid for large x because of the inequality (69). It is consistent to invoke the additional constraint

$$ky^2/x \ll 1 \tag{74}$$

under which

$$A(x,y) \approx 1 + \gamma\left[\frac{2}{\pi}\left(\frac{\pi x}{2k}\right)^{\frac{1}{2}} e^{i\pi/4} - y\right] \tag{75}$$

This compares with Wenzel's Eq. (47)

$$G \approx \left\{1 + \gamma_0\left[\left(\frac{\pi r}{2k}\right)^{\frac{1}{2}} e^{i\pi/4} - h\right]\right\}\frac{e^{ikr}}{2\pi r} \tag{76}$$

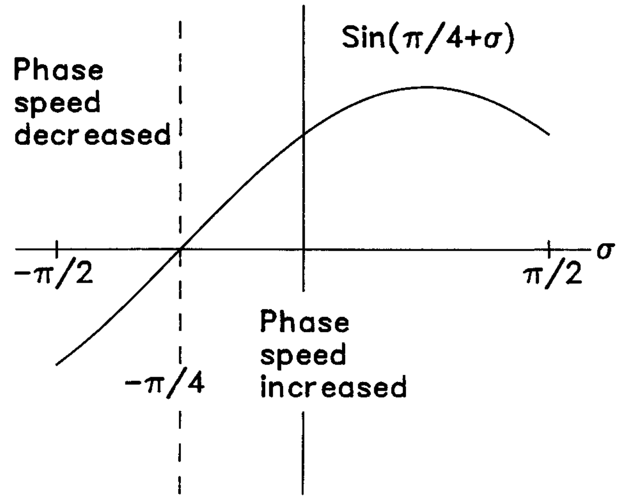
obtained under constraints equivalent to the inequalities (69) and (74). Again, the factor $\exp[ikx/(1+M)]$ is suppressed in the result given by Eq. (75). The conditions expressed in the inequalities (69) allow the approximation of Eq. (75) to be written as

$$A(x,y)\exp[ikx/(1+M)] \approx \exp(ik^*x) \tag{77}$$

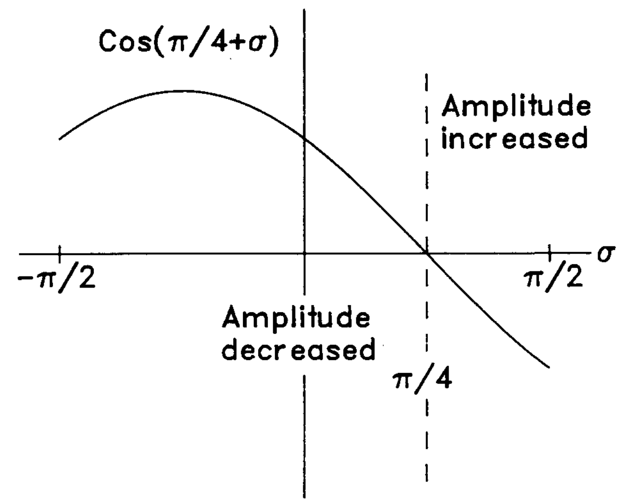
with

$$k^* = \left(\frac{k}{1+M}\right) + \gamma\left[\left(\frac{2}{\pi k x}\right)^{\frac{1}{2}} e^{-i\pi/4} + \frac{iy}{x}\right] \tag{78}$$

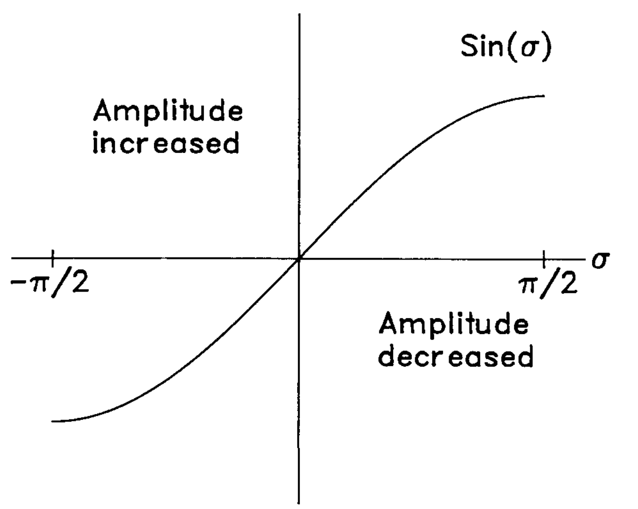
so that the x and y components of the phase velocity are then given by



a) The x component of the phase velocity relative to the rigid-boundary result as governed by the sign of $\sin(\pi/4+\sigma)$



b) Amplitude of the wave relative to the rigid-boundary result for fixed y as governed by the sign of $\cos(\pi/4+\sigma)$



c) Amplitude of the wave relative to the rigid-boundary result for fixed x as governed by the sign of $\sin(\sigma)$

Fig. 2 Summary of results for the hard boundary surface wave.

$$V_x = \frac{\omega}{|\gamma|}\left\{\frac{(1+M)|\gamma|(2\pi kx)^{\frac{1}{2}}}{k+|\gamma|(1+M)\sin(\pi/4+\sigma)}\right\} \tag{79}$$

$$V_y = \frac{-\omega}{|\gamma|\cos\sigma} \tag{80}$$

respectively. Since $\cos\sigma > 0$, this last equation implies a phase velocity directed toward the boundary.

The x component of the phase velocity has the following properties that are governed by the algebraic sign of the factor $\sin(\pi/4+\sigma)$; see Fig. 2a. For $-\pi/2 < \sigma < -\pi/4$, it is decreased relative to the rigid-boundary case, while for $-\pi/4 < \sigma < \pi/2$, it is increased. The magnitude of A, as given by Eq. (77), is

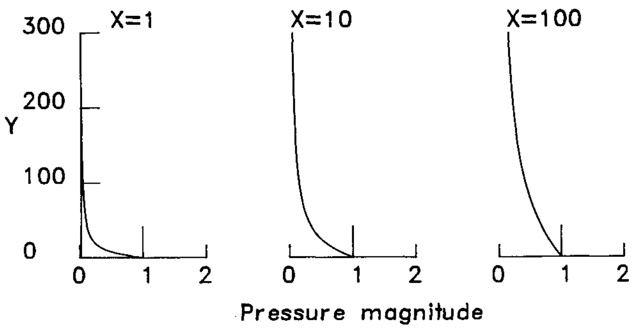
$$|A| = \exp\{-|\gamma|[(2x/k\pi)^{\frac{1}{2}}\cos(\pi/4+\sigma) + y\sin\sigma]\} \tag{81}$$

Thus, the amplitude of the wave, for fixed y, is decreased for $-\pi/2 < \sigma < \pi/4$, but increased for $\pi/4 < \sigma < \pi/2$ relative to the rigid-boundary case; see Fig. 2b. For fixed x, the wave is amplified for increasing y if $-\pi/2 < \sigma < 0$ and it decays with y for $0 < \sigma < \pi/2$; see Fig. 2c. These results are in complete agreement with those presented by Wenzel.

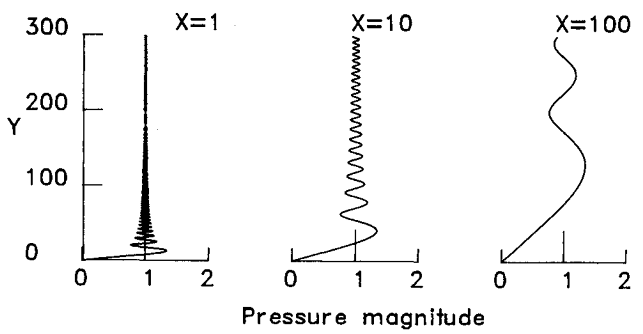
Illustrative Examples

The contribution of each of the terms of Eq. (45) to the total field will now be illustrated by consideration of specific examples. A degree of generality is maintained, however, by use of the scaled variables defined in Eqs. (39) and (40). The second term of the solution is independent of the boundary properties; the magnitude of this term, for three X locations, as a function of Y is shown in Fig. 3a. Note that this term is unity at $Y = 0$ and that it decays smoothly with increasing Y, approaching zero quite rapidly for all cases shown. The phase of this term is not shown here; however, a plot of its imaginary part vs its real part produces a curve reminiscent of Cornu's spiral. The interaction of this term with the incident wave is illustrated in Fig. 3b, which shows the magnitude of the first two terms of Eq. (45) for the same field points as used in the previous figure. The resulting diffraction bands are evident on the figure.

The effects of the phase angle of the impedance are illustrated in Fig. 4. The magnitudes of the sum of the last two terms of Eq. (45) for $\sigma = \pi/2$, $\pi/4$, and 0 are displayed in Figs. 4a, 4b, and 4c, respectively. Negative phase angles lead to results essentially the same as those in Fig. 4c. Again, these terms generally decay smoothly to zero with increasing Y. Note, however, the contribution of the surface wave, which is clearly shown in Fig. 4a at $X = 100$ and $Y \approx 0$. The smooth behavior of the magnitude of these two terms may be misleading because the phase of these terms varies quite rapidly. The total field, which is the result of interaction between the radiated and incident fields, exhibits the characteristic diffraction pattern shown in Fig. 5. In accordance with Eq. (62), a clear pressure minimum at a point above the boundary is evident for positive values of σ and no such minimum occurs for $\sigma \leq 0$.

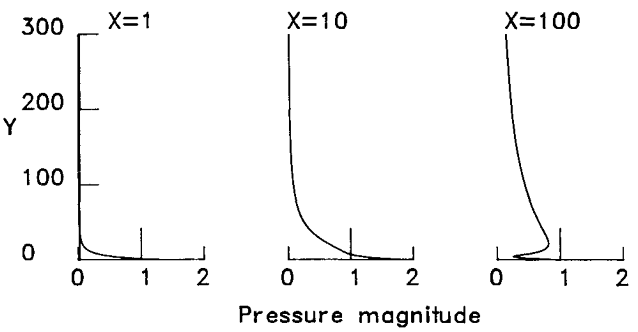


a) Impedance independent part of radiated wave

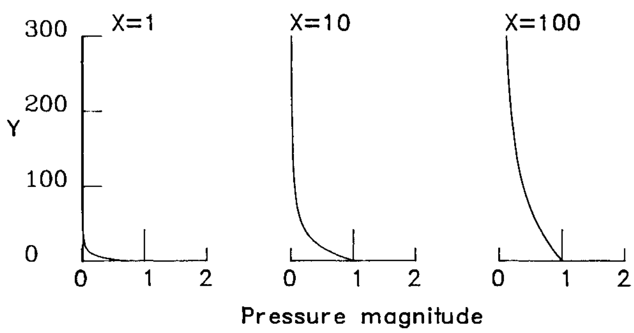


b) Incident wave plus impedance independent part of radiated wave

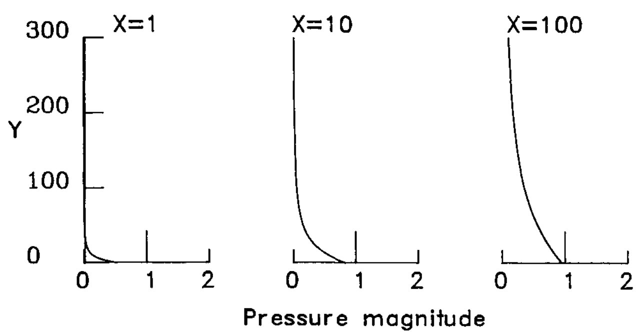
Fig. 3 Magnitude of the components of the total field relative to the magnitude of the incident wave.



a) $\sigma = \pi/2$



b) $\sigma = \pi/4$



c) $\sigma = 0$

Fig. 4 Magnitude of the radiated wave relative to the magnitude of the incident wave.

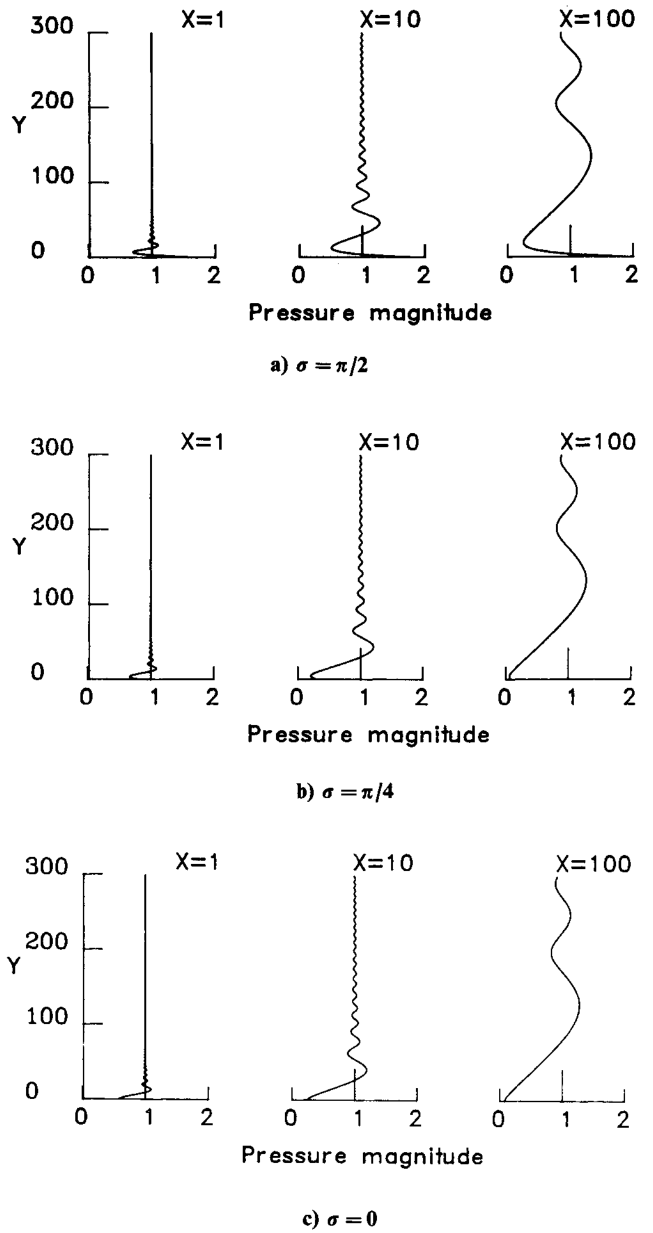


Fig. 5 Magnitude of the total field relative to the magnitude of the incident wave.

Conclusions

In this paper, an analytical solution has been obtained for the parabolic approximation describing the acoustic field of a plane wave at grazing incidence to a finite impedance boundary. Asymptotic expansion of this solution for both hard- and soft-boundary limits reproduces results obtained by Wenzel for a point source near an absorbing boundary, with the exception of effects that are solely a result of the different geometries considered. This implies that curvature of the wave fronts of the incident field may not be as important a factor as has previously been assumed when considering propagation at grazing incidence. Finally, the relative simplicity of the current analysis, when compared to that for the fully three-dimensional case,[9,16] suggests that the problem considered here may serve as a useful example for understanding of the effects of a finite impedance surface on disturbances at grazing incidence.

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